The electric field of a single photon

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***Abstract*—It is well known that the expectation value of the electric field operator in a number (Fock) state vanishes. However, a measurement of the electric field must yield some nonzero value each time an experiment is performed. What values can be obtained, and with what probability, is perhaps less well known. In this work we determine the probability distribution for the electric-field amplitude of an optical beam prepared in a single-photon state, using some techniques of quantum field theory. For the purpose of illustration, we apply our theory to the analysis of an ideal experiment where two spatially separated detectors can measure either the amplitude of the electric field, or the intensity of a single-photon optical beam. We find that while the photon cannot be detected in two disjoint positions at the same time, a measurement of the amplitude of the electric field carried by ditto photon, would yield a nonzero value at both detectors. These results show that a quantum field is both local with respect to its discrete excitations, i.e., the particles, and nonlocal with respect to its continuous amplitude.**

***Keywords*—single-photon states, field detection, electromagnetic field eigenstates, quadratures**

I. Introduction

In modern optics discrete and continuous excitations of a radiation field are customarily described by Fock and coherent states, respectively [1]. Fock states are eigenstates of the photon-number operator, and coherent states are eigenstates of the positive-frequency-component of the electric field operator [2]. However, little or no attention is usually paid to the eigenstates of the electric field operator. A few notable exceptions are the books by Schleich [3], Parker [4] and Barnett&Radmore [5], where the analogy between the single-mode electric field operator and the quadratures of a harmonic oscillator is exploited to infer the form of the electric field eigenstates. Vice versa, eigenstates of quantum field operators are more frequently considered in the path integral formulation of quantum field theory (QFT) [6]. In a beam of light prepared in an eigenstate of the electric field operator, the latter has a very well defined amplitude which can be, in principle, measured (see, e.g., [7], [8] and §9 of [9] for a thorough discussion about measurements of the strength of a quantum field). Conversely, if the light beam is not in an eigenstate of the electric field operator, repeated measurements of the field amplitude will yield different values distributed according to some probability density function. For any given quantum state of the light, such a function can be calculated using some standard techniques of QFT [10] and probability theory [11].

In this work we employ these techniques to calculate the probability distribution of the electric field amplitude of a paraxial beam of light prepared in a single-photon state. This allow us to show in a direct manner that a quantum optical field possesses both local and nonlocal characteristics. Specifically, we demonstrate that *discrete* excitations of the field, namely the photons, satisfy a locality condition (see (14)). Vice versa, *continuous* amplitudes of the field manifest nonlocal aspects (see (13)).

II. Quantum field theory of light

A. Paraxial quantum field operators

Following closely [12], we consider a monochromatic paraxial beam of light of frequency ω, propagating in the z direction and polarized along the x axis of a given Cartesian coordinate system. In the Coulomb gauge, the electric field operator can be written as $\hat{\mathbf{E}}(\mathbf{r},t) = \hat{\Phi}(\mathbf{r},t)\,\hat{\mathbf{e}}_x$, where $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ is the position vector and, in suitably chosen units,

$$\hat{\Phi}(\mathbf{r},t) = \left[e^{-i\omega t}\hat{\phi}(\mathbf{x},z) + e^{i\omega t}\hat{\phi}^\dagger(\mathbf{x},z)\right]/\sqrt{2}, \tag{1}$$

with $\hat{\phi}(\mathbf{x},z) = \sum_\mu \hat{a}_\mu u_\mu(\mathbf{x},z)$, $\mathbf{x} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$ the transverse position vector, and μ the mode index. The annihilation and creation operators $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$, respectively, satisfy the bosonic canonical commutation relations $\left[\hat{a}_\mu,\ \hat{a}_{\mu'}^\dagger\right] = \delta_{\mu\mu'}$. By hypothesis, the paraxial modes $u_\mu(\mathbf{x},z)$ form a complete and orthogonal set of basis functions on $\mathbb{R}^2$, i.e., $\sum_\mu u_\mu(\mathbf{x},z)u_\mu^*(\mathbf{x}',z) = \delta(\mathbf{x}-\mathbf{x}')$, and $\big(u_\mu, u_{\mu'}\big) = \delta_{\mu\mu'}$, respectively. Here and hereafter we use the suggestive notation $(f,g) = \int \mathrm{d}^2x\, f^*(\mathbf{x},z)g(\mathbf{x},z)$. We remark that throughout this paper the transverse coordinates x and y are dynamical variables, while the longitudinal coordinate z is a parameter [12].

B. Quantum states of the electromagnetic field

Consider a classical light beam carrying the electric field $\mathbf{E}_{\mathrm{cl}}(\mathbf{r},t) = \Phi(\mathbf{r},t)\,\hat{\mathbf{e}}_x$, where $\Phi(\mathbf{r},t) = \left[e^{-i\omega t}\phi(\mathbf{x},z) + e^{i\omega t}\phi^\dagger(\mathbf{x},z)\right]/\sqrt{2}$, with $\phi(\mathbf{x},z)$ normalized to $(\phi,\phi) = 1$. By construction, the classical field $\Phi(\mathbf{r},t)$ is equal to the expectation value of the quantum field $\hat{\Phi}(\mathbf{r},t)$ with respect to the coherent state $|\{\phi\}\rangle$, i.e., $\Phi(\mathbf{r},t) = \langle\{\phi\}|\hat{\Phi}(\mathbf{r},t)|\{\phi\}\rangle$, where $|\{\phi\}\rangle = \exp\!\big(\hat{a}^\dagger[\phi] - \hat{a}[\phi]\big)|0\rangle$, $|0\rangle$ is the vacuum state of the electromagnetic field defined by $\hat{a}_\mu|0\rangle = 0$ for all μ, and $\hat{a}^\dagger[\phi] = \sum_\mu \hat{a}_\mu^\dagger\phi_\mu$, with $\phi_\mu = (u_\mu,\phi)$ [2], [12]. It is not difficult to show that $\left[\hat{a}[\phi], \hat{a}^\dagger[\psi]\right] = \big(\phi,\psi\big)$, for any pair of normalized fields $\phi(\mathbf{x},z)$ and $\psi(\mathbf{x},z)$. The field $\phi(\mathbf{x},z)$ also determines the (improperly) so-called wave function of the photon, defined by $\langle 0|\hat{\Phi}(\mathbf{r},t)|1[\phi]\rangle = e^{-i\omega t}\phi(\mathbf{x},z)/\sqrt{2}$, where $|N[\phi]\rangle = \left(\hat{a}^\dagger[\phi]\right)^N|0\rangle/\sqrt{N!}$, denotes the N-photon Fock state with $N = 0, 1, 2, \ldots$, such that $\hat{N}|N[\phi]\rangle = N|N[\phi]\rangle$, $\hat{N} = \sum_\mu \hat{a}^\dagger_\mu \hat{a}_\mu$, and $\langle N[\phi]|M[\psi]\rangle = \left(\phi,\psi\right)^N \delta_{NM}$.

C. Quantum fields as operator valued distributions

In QFT, an object like $\hat{\Phi}(\mathbf{x},z,t)$ is not really a proper observable, i.e., an Hermitian operator in the Hilbert space $\mathcal{H}$ of the physical states, but rather an "operator valued distribution" over the Euclidean spacetime $\mathbb{R}^2\times\mathbb{R}$ [13]. To obtain a bona fide Hermitian operator defined on the vectors in $\mathcal{H}$, we must to smear out $\hat{\Phi}(\mathbf{x},z,t)$ with a smooth (infinitely differentiable) test function $f(\mathbf{x},t) \in \mathbb{R}$ [14], [15], namely to take $\hat{\Phi}[f] = \int \mathrm{d}^2x\, \mathrm{d}t\, f(\mathbf{x},t)\hat{\Phi}(\mathbf{x},z,t)$. In the case of free fields, we can harmlessly choose $f(\mathbf{x},t) = \delta(t)f(\mathbf{x})$ [13], [16], so that

$$\begin{aligned}\hat{\Phi}[f] &= \int \mathrm{d}^2x\, f(\mathbf{x})\hat{\Phi}(\mathbf{x},z,0)\\ &= \sum_\mu \left(\hat{a}_\mu f^*_\mu + \hat{a}^\dagger_\mu f_\mu\right)/\sqrt{2},\end{aligned} \tag{2}$$

where $f_\mu = (u_\mu, f) = |f_\mu|e^{i\lambda_\mu}$ [13], [16]. For example, $f(\mathbf{x})$ can be a Gaussian function strongly peaked near some point $\mathbf{x}_0$, so that $\hat{\Phi}[f]$ gives an estimate of $\hat{\Phi}(\mathbf{x}_0,z,0)$. Note that we can rewrite $\hat{\Phi}[f] = \sum_\mu |f_\mu|\,\hat{x}_{\lambda_\mu}$, where $\hat{x}_{\lambda_\mu} = \left(\hat{a}_\mu e^{-i\lambda_\mu} + \hat{a}^\dagger_\mu e^{i\lambda_\mu}\right)/\sqrt{2}$ is the quadrature operator of the field component with respect to the mode u_μ [5].

In the remainder we will also consider the "intensity" operator valued distribution $\hat{\mathrm{I}}(\mathbf{x},z)$ defined by $\hat{\mathrm{I}}(\mathbf{x},z) = \hat{\phi}^\dagger(\mathbf{x},z)\hat{\phi}(\mathbf{x},z)$. This can be interpreted as a photon-number operator per unit transverse surface, because $\int_{\mathbb{R}^2} \mathrm{d}^2x\,\hat{\mathrm{I}}(\mathbf{x},z) = \hat{N}$. Let $\mathcal{A}$ be the area of the active surface of a photodetector placed somewhere in the transverse plane at z fixed. With $F(\mathbf{x}) \geq 0$ we denote a test function normalized to $\int \mathrm{d}^2x\, F(\mathbf{x}) = \mathcal{A}$, whose compact support coincides with the detector active surface. Then, we define the smeared intensity operator $\hat{\mathrm{I}}[F]$ as

$$\begin{aligned}\hat{\mathrm{I}}[F] &= \int \mathrm{d}^2x\, F(\mathbf{x})\,\hat{\mathrm{I}}(\mathbf{x},z)\\ &= \sum_{\mu,\nu} \hat{a}^\dagger_\mu \hat{a}_\nu F_{\mu\nu},\end{aligned} \tag{3}$$

where $F_{\mu\nu} = (u_\mu, F u_\nu)$.

III. Probability distributions

In probability theory, the probability density function (p.d.f.) $p_Q(\mathbf{q})$ of a D-dimensional multivariate random variable $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_D)$, can be written as $p_Q(\mathbf{q}) = \langle \delta(\mathbf{Q}-\mathbf{q})\rangle$, where $\langle\cdots\rangle$ denotes average over all possible realization of $\mathbf{Q}$, and $\delta(\mathbf{Q}-\mathbf{q}) = \prod_{n=1}^D \delta(Q_n - q_n)$ [11]. Similarly, in QFT given a Hermitian operator $\hat{K}$, and a vector state $|\psi\rangle$ of norm 1, we can equivalently calculate the expectation value of any regular function $F(\hat{K})$ of $\hat{K}$ with respect to $|\psi\rangle$, either as $\langle F(\hat{K})\rangle = \langle\psi|F(\hat{K})|\psi\rangle$, or as $\langle F(\hat{K})\rangle = \int \mathrm{d}k\, F(k)p_\psi(k)$, where the p.d.f. $p_\psi(k)$ is defined by $p_\psi(k) = \langle\psi|\delta\big(\hat{K}-k\big)|\psi\rangle$ (see, e.g., sec. 3-1-2 in [16], or problem 4.3 in [10]). Using the Fourier representation of the Dirac delta function $\delta(z) = \int_{\mathbb{R}} \mathrm{d}\alpha\, e^{i\alpha z}/(2\pi)$, it is straightforward to show that

$$p_\psi(k) = \int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{2\pi}\, e^{-i\alpha k}\langle\psi|e^{i\alpha\hat{K}}|\psi\rangle, \tag{4}$$

where $\langle\psi|\exp(i\alpha\hat{K})|\psi\rangle$ is the so-called quantum characteristic function [17].

IV. Local and nonlocal aspects of quantum fields

Consider two experiments where D detectors are placed at spatially separated points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_D$ in a transverse plane at fixed z. In the first experiment each detector measures the amplitude A_n of the field $\hat{\Phi}_n \equiv \hat{\Phi}[f_n]$, where $f_n(\mathbf{x}) \equiv f(\mathbf{x}-\mathbf{x}_n)$, and $n = 1, \ldots, D$. In the second experiment each detector measures the value I_n of the intensity operator $\hat{\mathrm{I}}_n \equiv \hat{\mathrm{I}}[F_n]$, where $F_n(\mathbf{x}) \equiv F(\mathbf{x}-\mathbf{x}_n)$, with $F_n(\mathbf{x})F_m(\mathbf{x}) = 0$ for $n \neq m$ (disjoint supports). From Sec. III, it follows that the field random amplitudes $\mathbf{A} = (A_1, A_2, \ldots, A_D)$, are distributed according to

$$p_N(\mathbf{A}) = \langle N[\phi]|\prod_{n=1}^D \delta\big(\hat{\Phi}_n - A_n\big)|N[\phi]\rangle. \tag{5}$$

Similarly, the distribution of the intensity random values $\mathbf{I} = (I_1, I_2, \ldots, I_D)$, follows the p.d.f.,

$$p_N(\mathbf{I}) = \langle N[\phi]|\prod_{n=1}^D \delta\big(\hat{\mathrm{I}}_n - I_n\big)|N[\phi]\rangle. \tag{6}$$

Note that there is not an operator ordering problem in the definitions (5) and (6) because, by construction, $\big[\hat{\Phi}_n, \hat{\Phi}_m\big] = 0 = \big[\hat{\mathrm{I}}_n, \hat{\mathrm{I}}_m\big]$, for all $n, m = 1, 2, \ldots, D$.

A. Single detector: $D = 1$

In the simplest case of vacuum state ($N = 0$) and a single detector ($D = 1$), we recover the textbooks results

$$p_0(A) = \left(2\pi\sigma^2\right)^{-1/2}\exp\left(-\frac{A^2}{2\sigma^2}\right), \tag{7}$$

$$p_0(I) = \delta(I), \tag{8}$$

where $\sigma = (f,f)/\sqrt{2}$. Note that $p_0(I) = \delta(I)$ signifies that the probability to measure a nonzero intensity ($I \neq 0$) in the vacuum state is zero.

Next, we consider the probability distributions with respect to the single-photon state ($N = 1$). A straightforward calculation gives

$$p_1(A) = p_0(A)\left[\left(1-|s|^2\right) + 2\left|A\,\frac{s}{\|f\|}\right|^2\right], \tag{9}$$

$$p_1(I) = \Big(\phi, \delta\big(F(\mathbf{x}) - I\big)\phi\Big), \tag{10}$$

where $\|f\|^2 = (f,f)$, $s = (\phi,f)/\|f\|$, and the Cauchy–Schwarz inequality guarantees that $1-|s|^2 \geq 0$. Equation (9) shows that the value of $p_1(A)$ is completely determined by the superposition s between the wave function of the photon ϕ and the test function f, which characterizes the detector.

In the expression of $p_1(I)$, the argument of the delta function implicitly defines a plane curve of equation $F(x, y) - I = 0$, which can be solved, for example, with respect to y. Then, applying the well-known formula for a delta of a function, one can easily evaluate $p_1(I)$.

B. Two detectors: $D = 2$

For the vacuum state a trivial calculation gives $p_0(\mathbf{A}) = p_0(A_1)p_0(A_2)$ and $p_0(\mathbf{I}) = p_0(I_1)p_0(I_2)$. For $N = 1$, a little calculation shows that

$$p_1(\mathbf{A}) = p_0(\mathbf{A}) \Bigg[\big(1 - |s_1|^2 - |s_2|^2\big) + 2 \left| A_1 \frac{s_1}{\|f_1\|} + A_2 \frac{s_2}{\|f_2\|} \right|^2 \Bigg], \tag{11}$$

$$p_1(\mathbf{I}) = \Big(\phi, \delta\big(F_1(\mathbf{x}) - I_1\big)\delta\big(F_2(\mathbf{x}) - I_2\big)\phi\Big), \tag{12}$$

where $\|f_n\|^2 = (f_n, f_n)$, $s_n = (\phi, f_n)/\|f_n\|$, with $n = 1, 2$. Both p.d.f.s are correctly normalized to "1". Let us analyze $p_1(\mathbf{A})$ first. To this end, it is instructive to rewrite (11) as

$$p_1(\mathbf{A}) = p_1(A_1)p_0(A_2) + p_0(A_1)p_1(A_2) + p_0(\mathbf{A}) \left[2 \frac{A_1 A_2}{\|f_1\| \, \|f_2\|} \, (s_1 s_2^* + s_1^* s_2) - 1 \right], \tag{13}$$

For $A_1 \neq 0$ and $A_2 \neq 0$ the term proportional to $p_0(\mathbf{A}) = p_0(A_1)p_0(A_2)$ become negligible and $p_1(\mathbf{A}) \approx p_1(A_1)p_0(A_2) + p_0(A_1)p_1(A_2)$. This means that two contributions to $p_1(\mathbf{A})$ are due to the photon nearby $\mathbf{x}_1$ (via the term $p_1(A_1)p_0(A_2)$) and to the photon nearby $\mathbf{x}_2$ (via $p_0(A_1)p_1(A_2)$. However, as typically both terms do not vanish, we have a nonzero probability to detect the field amplitude A_1 near $\mathbf{x}_1$ and the field amplitude A_2 near $\mathbf{x}_2$. This demonstrates the nonlocality of the electromagnetic field in the single-photon state.

We pass now to study $p_1(\mathbf{I})$. Let $\mathcal{C}_2$ be the plane curve where the equation $F_2(\mathbf{x}) - I_2 = 0$ is satisfied. By definition, $\mathcal{C}_2$ is contained within the support of $F_2(\mathbf{x})$, where $F_1(\mathbf{x}) = 0$ because, by hypothesis, $F_1(\mathbf{x})F_2(\mathbf{x}) = 0$. This implies that $\delta\big(F_1(\mathbf{x}) - I_1\big) = \delta\big(I_1\big)$ for all points in $\mathcal{C}_2$. Therefore, in this case

$$p_1(\mathbf{I}) = \delta(I_1)p_1(I_2) = p_0(I_1)p_1(I_2). \tag{14}$$

Vice versa, if $F_1(\mathbf{x}) - I_1 = 0$ holds, then by the same reasoning we obtain $p_1(\mathbf{I}) = p_1(I_1)p_0(I_2)$. The latter two equations tell us that in each run of the experiment, only one of the two detectors can fire. This is a manifestation of the locality of the discrete excitations of the field.

V. Conclusions

In this work we have studied the problem of locality and nonlocality in quantum electromagnetic fields. Unlike more conventional approaches (see, e.g, the seminal work by Tan *et al.*, [18]), we have used mathematical methods that are familiar from quantum field theory and probability theory. However, apart from the use of different techniques, our results are in agreement and confirm the ones presented in [18]. Specifically, we have found that a light beam prepared in a single-photon state can manifest both local and nonlocal features according as to whether the intensity or the amplitude of the field is measured. Our approach based on quantum field theory is perfectly general and allows fields and particles to be given equal weight. We believe that this work can promote the use of these techniques in various applications of quantum optics.

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